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Unique solutions to boundary value problems in the cold plasma model

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Abstract

The unique existence of a weak solution to the homogeneous closed Dirichlet problem on certain D-star-shaped domains is proven for a mixed elliptic-hyperbolic equation. Equations of this kind arise in models for electromagnetic wave propagation in cold plasma. A related class of open boundary value problems is shown to possess strong solutions.

Key words. Equations of mixed elliptic-hyperbolic type, cold plasma model, closed boundary value problem, symmetric positive operator, Cinquini-Cibrario equation

AMS subject classifications. 35M10, 35D30, 82D10

1 Introduction

Boundary value problems for mixed elliptic-hyperbolic equations may be either *open* or *closed*. In the former case, data are prescribed on a proper subset of the boundary whereas in the latter case, data are prescribed on the entire boundary. It is shown in Sec. 3 of [22] that if $\kappa = 1/2$, the closed Dirichlet problem is over-determined for the equation

$$\left(x - y^2\right) u_{xx} + u_{yy} + \kappa u_x = 0 \tag{1.1}$$

on a typical domain, where $u(x, y)$ is required to be twice-continuously differentiable on the domain. However, this equation arises in a qualitative model

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for electromagnetic wave propagation in an idealized cold plasma ([39], eq. 81; see also [30], eq. (9)). Physical reasoning suggests that the closed Dirichlet problem for (1.1) should be well-posed in a suitable function space, at least for some choice of lower order terms; so the result reported in [22] would appear to represent a serious deficiency in the physical model. See Sec. 1 of [22] for a discussion, in which the problem of formulating a closed Dirichlet-like problem that is well-posed in an appropriate sense is characterized as an "outstanding and significant problem for the cold plasma model."

Except for the point at the origin, eq. (1.1) can be transformed into an equation of Tricomi type; see, *e.g.*, Sec. 2.4.3 of [26]. Such equations have somewhat more desirable analytic properties than eq. (1.1). However, both the physical and mathematical interest of eq. (1.1) arise from the tangency of the resonance curve to a flux line at the origin. This is the point at which plasma heating might occur in the physical model, and a point which appears to be singular in studies of the solutions to (1.1); see [22], [30], and [39]. Thus it is not generally useful to transform eq. (1.1) into an equation of Tricomi type.

Using methods introduced by Lupo, Morawetz, and Payne [17], [18] for equations of Tricomi type, we show in Sec. 2 the weak existence of a unique solution to a homogeneous closed Dirichlet problem for the formally self-adjoint ($\kappa = 1$) case of eq. (1.1). This extends a recent result [24] in which the existence of solutions having various degrees of smoothness was shown in certain cases to which uniqueness proofs did not seem to apply. At the same time, it extends the unique-existence arguments in [17] to an equation which is not of Tricomi type.

Another well known problem in elliptic-hyperbolic theory is the determination of natural conditions for boundary geometry; see the discussions in [1], [20], [21], [27], [28], and [29]. Heuristic approaches to determining boundary geometry tend to focus on physical [19] or geometric [25] analogies for the specific equation under study. In his theory of symmetric positive systems [8], Friedrichs proposed intrinsic mathematical criteria for the well-posedness, or *admissibility*, of boundary conditions. But Friedrichs' conditions are also tied to the specifics of the particular symmetric positive equation under study and are algebraic rather than explicitly geometric. We will require boundary arcs to be starlike with respect to an appropriate vector field. This approach to boundary geometry was introduced by Lupo and Payne [16]. We note that algebraic conditions in certain very old results can be reinterpreted as the requirement of a starlike boundary; see, *e.g.*, [11]. Our results provide further

evidence that domain boundaries which are starlike in this generalized sense are natural for elliptic-hyperbolic boundary value problems.

In Sec. 3 we investigate the solvability of open boundary value problems for a class of symmetric positive systems on domains having starlike boundaries. This class includes an equation originally studied by Cinquini-Cibrario in the early 1930s. The equation arises in the so-cold "slab" model of zero-temperature plasma and in models of low-temperature atmospheric and space plasmas (Sec. 3.1).

The boundary conditions in Sec. 3 are *mixed* in the sense that a Dirichlet condition is placed on part of the boundary and a Neumann condition is placed on another part. However, our methods also apply to the case in which either a Dirichlet or a Neumann condition is imposed over the entire elliptic boundary. Because the boundary value problems considered in Sec. 3 are open, the results of that section may be less interesting physically than those of Sec. 2. But open boundary conditions can be expected to imply more smoothness on the part of solutions than is obtained from closed boundary conditions, and we provide conditions for the existence of solutions which are strong in the sense of Friedrichs. Conditions for the existence of strong solutions to elliptic-hyperbolic boundary value problems were also discussed in Sec. 3 of [24], but briefly and inadequately. Section 3 of this report revises and extends (to the open case of mixed and Neumann problems) the treatment of strong solutions in [24]. The existence question for weak solutions to open boundary value problems for equations of the form (1.1) was considered in [23] and [41].

1.1 Remarks on the physical model

In the cold plasma model, the plasma temperature is assumed to be zero in order to neglect the fluid properties of the medium, which is treated as a linear dielectric. In the case of wave propagation through an underlying static medium having axisymmetric geometry, equations of the form (1.1) model the tangency of a flux surface to a resonance surface. At the point of tangency, plasma heating might occur even in the cold plasma model [39]. In two dimensions, flux surfaces (level sets of the magnetic flux function) can be represented by the lines $x = \text{const.}$, and a resonance surface (frequencies at which the field equations change from elliptic to hyperbolic type) by the curve $x = y^2$. In such cases a plasma heating zone could lie at the origin of coordinates. This conjecture is supported by numerical [22] and classical [30]

analysis which suggests that the origin is a singular point of eq. (1.1).

The main physical implication of our result is that, although the Dirichlet problem for the cold plasma model is manifestly ill-posed in the classical sense, there is a weak sense in which the closed Dirichlet problem is no longer over-determined: a unique weak solution to the closed Dirichlet problem exists in an appropriately weighted function space. The weight function vanishes on the resonance curve. In Corollary 6 of [24] an existence theorem, without uniqueness, was demonstrated in a weighted function space in which the weight function vanished on the line $y = 0$. Thus in each case the weight function vanishes at the origin of coordinates. These results support the physical conjecture of a heating zone at that point.

For discussions of the physical context of eq. (1.1), see Sec. V of [39] and Sec. 4 of [40], in which eq. (1.1) with $\kappa = 0$ is proposed as a qualitative model for erratic heating effects by lower hybrid waves in the plasma. See also [30], in which a model for electrostatic waves in a cold anisotropic plasma with a two-dimensional inhomogeneity yields, by a formal derivation, an equation for the field potential which is similar to (1.1). Briefly, the derivation proceeds as follows: at zero temperature, the field equations reduce to Maxwell's equations, which are written for the electric displacement vector $\mathbf{D}$. The components of $\mathbf{D}$ are written in terms of the dielectric tensor for the medium. The coordinate system is chosen so that, in two dimensions, the y-axis is collinear to a longitudinally applied magnetic field. In that case the dielectric tensor assumes a particular form which leads, when the resonance curve is tangent to the flux line at the origin, to an equation having the form considered here. Precisely, the equation derived in [30] is eq. (3.29) of Remark *iii)*, Sec. 3, below, with particular choices of u_1, u_2, $\sigma(y)$, κ_1, and κ_2.

The derivation of the governing equation for the case of fully electromagnetic waves is much more lengthy, and occupies much of the memoir [39]. In that case, equations having the form (1.1) provide only a qualitative description of the physical model. Section 2.5 of [26] is devoted to a brief derivation of eq. (1.1) for electromagnetic waves.

The classic text for wave propagation in plasma is [35]; see also [9]. See Ch. 2 of [36] for a concise introduction to the physics of electromagnetic waves in cold plasma, and [26] for a recent review of mathematical aspects of the cold plasma model.

In the sequel we assume that Ω is an open, bounded, connected domain of $\mathbb{R}^2$ having at least piecewise continuously differentiable boundary with counterclockwise orientation. For nontriviality we require that Ω contain an

arc of the resonance curve $x = y^2$ ($x = \sigma(y)$ in Sec. 3). Additional conditions will be placed on the domain where required.

2 Weak solutions to closed boundary value problems

Following Sec. 3 of [17] we define, for a given C^1 function $K(x,y)$, the space $L^2(\Omega; |K|)$ and its dual. These spaces consist, respectively, of functions u for which the norm

$$||u||_{L^2(\Omega;|K|)} = \left(\int\int_\Omega |K|u^2 dxdy\right)^{1/2}$$

is finite, and functions $u \in L^2(\Omega)$ for which the norm

$$||u||_{L^2(\Omega;|K|^{-1})} = \left(\int\int_\Omega |K|^{-1}u^2 dxdy\right)^{1/2}$$

is finite. Analogously, we define the space $H^1(\Omega; K)$ to be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$||u||_{H^1(\Omega;K)} = \left[\int\int_\Omega \left(|K|u_x^2 + u_y^2 + u^2\right)\, dxdy\right]^{1/2}, \tag{2.1}$$

and introduce the space $H_0^1(\Omega; K)$ as a closure in this space. Using a weighted Poincaré inequality to absorb the zeroth-order term, we write the $H_0^1(\Omega; K)$-norm in the form

$$||u||_{H_0^1(\Omega;K)} = \left[\int\int_\Omega \left(|K|u_x^2 + u_y^2\right)\, dxdy\right]^{1/2}. \tag{2.2}$$

The dual space $H^{-1}(\Omega; K)$ is defined via the negative norm

$$||w||_{H^{-1}(\Omega;K)} = \sup_{0\neq\varphi\in C_0^\infty(\Omega)} \frac{|\langle w, \varphi\rangle|}{||\varphi||_{H_0^1(\Omega,K)}},$$

where $\langle\ ,\ \rangle$ is the Lax duality bracket.

Various lower-order terms have been associated to eq. (1.1) in the literature on the cold plasma model; only the higher-order terms contribute to

resonance. These lower-order terms were considered explicitly in Sec. 2 of [24], in which we showed the existence of L^2 solutions to a closed Dirichlet problem for eq. (1.1). Here we consider only the formally self-adjoint case

$$Lu \equiv [K(x,y)\,u_x]_x + u_{yy} = f(x,y)\,, \tag{2.3}$$

for the type-change function $K(x,y) = x - y^2$. As in [24], the inhomogeneous term $f(x,y)$ is assumed known. For this special case of lower-order terms, we are able to show the existence of solutions to the closed Dirichlet problem having much higher regularity than was the case in [24].

The forcing function f in eq. (2.3) (and also in eq. (3.1), below) could arise physically under appropriate conditions on the charge density. In addition, the presence of f provides mathematical generality, avoids trivial solutions, and allows the conversion of solutions to the Dirichlet problem with homogeneous boundary conditions into solutions to a problem for the homogeneous equation with inhomogeneous boundary conditions. Finally, analysis of the inhomogeneous equation is useful when, as in our case, one expects singularities in the homogeneous equation.

In accordance with standard terminology, we will often refer to the curve $K = 0$ on which eq. (2.3) changes type as the *sonic curve.* This terminology is borrowed from fluid dynamics; in the context of the cold plasma model, the sonic transition occurs at a resonance frequency.

Following Lupo, Morawetz, and Payne [17], we define a *weak solution* of eq. (2.3) on Ω, with boundary condition

$$u(x,y) = 0\,\forall\,(x,y) \in \partial\Omega, \tag{2.4}$$

to be a function $u \in H_0^1(\Omega;K)$ such that $\forall \xi \in H_0^1(\Omega;K)$ we have

$$\langle Lu, \xi\rangle \equiv -\int\int_\Omega (Ku_x\xi_x + u_y\xi_y)\,dxdy = \langle f, \xi\rangle,$$

where $\langle\ ,\ \rangle$ is duality pairing between $H_0^1(\Omega, K)$ and $H^{-1}(\Omega, K)$. In this case the existence of a weak solution is equivalent to the existence of a sequence $u_n \in C_0^\infty(\Omega)$ such that

$$||u_n - u||_{H_0^1(\Omega;K)} \to 0 \text{ and } ||Lu_n - f||_{H^{-1}(\Omega;K)} \to 0$$

as n tends to infinity.

Following Lupo and Payne (Sec. 2 of [16]), we consider a one-parameter family $\psi_\lambda(x,y)$ of inhomogeneous dilations given by

$$\psi_\lambda(x,y) = \left(\lambda^{-\alpha}x, \lambda^{-\beta}y\right),$$

where $\alpha, \beta, \lambda \in \mathbb{R}^+$, and the associated family of operators

$$\Psi_\lambda u = u \circ \psi_\lambda \equiv u_\lambda.$$

Denote by D the vector field

$$Du = \left[\frac{d}{d\lambda}u_\lambda\right]_{|\lambda=1} = -\alpha x\partial_x - \beta y\partial_y. \tag{2.5}$$

An open set $\Omega \subseteq \mathbb{R}^2$ is said to be *star-shaped* with respect to the flow of D if $\forall\,(x_0,y_0) \in \overline{\Omega}$ and each $t \in [0,\infty]$ we have $F_t(x_0,y_0) \subset \overline{\Omega}$, where

$$F_t(x_0,y_0) = (x(t),y(t)) = \left(x_0e^{-\alpha t}, y_0e^{-\beta t}\right).$$

If a domain is star-shaped with respect to a vector field D, then it is possible to “float” from any point of the domain to the origin along the flow lines of the vector field. If these flow lines are straight lines through the origin $(\alpha = \beta)$, then we recover the conventional notion of a star-shaped domain. By an appropriate translation, the origin can be replaced by any point (x_s,y_s) in the plane as a source of the flow. In that case we obtain a translated function $\tilde{F}_t$ for which

$$\lim_{t\to\infty} \tilde{F}_t(x_0,y_0) = (x_s,y_s)\ \ \forall\,(x_0,y_0) \in \overline{\Omega}.$$

Moreover, whenever a domain is star-shaped with respect to the flow of a vector field satisfying (2.5), the domain boundary will be *starlike* in the sense that

$$(\alpha x, \beta y) \cdot \hat{\mathbf{n}}(x,y) \geq 0,$$

where $\hat{\mathbf{n}}$ is the outward-pointing normal vector on the boundary $\partial\Omega$. See Lemma 2.2 of [16]. In equivalent notation, given a vector field $V = -(b,c)$ and a boundary arc Γ which is starlike with respect to V, the inequality

$$bn_1 + cn_2 \geq 0 \tag{2.6}$$

is satisfied on Γ.

2.1 An auxiliary problem

We employ the *dual variational method*, an integral variant of the *abc* method, introduced by Didenko [7] and developed by Lupo and Payne [15]. (Note that the term "dual variational" also appears in elliptic variational theory, in which it means something entirely different.) Denote by v a solution to the boundary value problem

$$Hv = u \text{ in } \Omega \tag{2.7}$$

for $u \in C_0^\infty(\Omega)$, with v vanishing on $\partial\Omega\backslash\{(0,0)\}$,

$$\lim_{(x,y)\to(0,0)} v\,(x,y) = 0, \tag{2.8}$$

and

$$Hv = av + bv_x + cv_y. \tag{2.9}$$

Assume that Ω is star-shaped with respect to the flow of the vector field $V = -\,(b,c)\,;\; b = mx$ and $c = \mu y$; μ and m are positive constants and a is a negative constant; the point $(x,y) = (0,0)$ lies on $\partial\Omega$. Step 1 in the proof of Lemma 3.3 in [17], which treats the harder case of a non-differentiable coefficient, demonstrates that v satisfying (2.7)-(2.9) exists and lies in the space $C^0\left(\overline{\Omega}\right) \cap H_0^1\left(\Omega; K\right)$. The proof is straightforward in our case, except perhaps for the justification of (2.8), which we outline:

Because Ω is star-shaped with respect to $-\,(b,c)$, any flow line entering the interior of Ω from $\partial\Omega$ will remain in $\overline{\Omega}$. So the method of characteristics yields a smooth solution to (2.7), (2.9), with a possible singularity at the origin. It remains only to analyze the limiting behavior of the solution near the origin. Because the origin lies on the boundary and u has compact support, we can restrict our attention to an ε-neighborhood N_ε of the boundary,

$$N_\varepsilon\left(\partial\Omega\right) = \left\{(x,y) \in \overline{\Omega} | \operatorname{dist}\left((x,y)\,,\partial\Omega\right) \le \varepsilon\right\},$$

where ε is so small that N_ε lies in $\overline{\Omega}$ but outside the support of u in Ω. In this subdomain we solve the Cauchy problem for the homogeneous equation

$$mxv_x + \mu y v_y = |a|v. \tag{2.10}$$

The characteristic equation for (2.10), given by $mxdy = \mu ydx$, can be integrated to yield

$$y = cx^{\mu/m}, \tag{2.11}$$

where c is a constant. The method of characteristics thus yields a general solution in N_ε having the form

$$v(x,y) = \varphi\left(\frac{x^\mu}{y^m}\right) y^{|a|/\mu}, \tag{2.12}$$

where φ is an arbitrary C^1 function that may be prescribed along a non-characteristic curve in N_ε. Following the evolution of the solution (2.12) along the curves (2.11), we obtain (2.8) as in eq. (3.13) of [17].

Following Sec. 2 of [7] (see also the Appendix to [15]), the dependence of the solution v to the problem (2.7)-(2.9) on the forcing function u is represented by an operator $\mathcal{I}$, writing $v = \mathcal{I}u$. We have the integral identities

$$(\mathcal{I}u, Lu) = (v, Lu) = (v, LHv). \tag{2.13}$$

A good choice of the coefficients a, m, and μ in the operator H on the right-hand side of this identity will allow us to derive an energy inequality, which will be used to prove weak existence via the Riesz Representation Theorem; see Ch. 2 of [2] for a general treatment of such "projection" arguments.

2.2 Main result

The following is a small but crucial extension of [24], Theorem 5.

Lemma 2.1. *Suppose that x is non-negative on $\overline{\Omega}$ and that the origin of coordinates lies on $\partial\Omega$. Let Ω be star-shaped with respect to the flow of the vector field $V = -(b,c)$ for $b = mx$ and $c = \mu y$, where m and μ are positive constants and m exceeds 3μ. Then there exists a positive constant C for which the inequality*

$$||u||_{L^2(\Omega;|K|)} \leq C||Lu||_{H^{-1}(\Omega;K)}$$

holds for every $u \in C_0^\infty(\Omega)$, where $K(x,y) = x - y^2$ and L is defined by (2.3).

Proof. Let v satisfy eqs. (2.7)-(2.9) on Ω for $a = -M$, where M is a positive number satisfying

$$M = \frac{m-3\mu}{2} - \delta$$

for some sufficiently small positive number δ. Integrate the identities (2.13) by parts, using Prop. 12 of [24] and the compact support of u. We have

$$\int\int_\Omega v\cdot LHv\,dxdy=\frac{1}{2}\oint_{\partial\Omega}\left(Kv_x^2+v_y^2\right)(cdx-bdy)$$
$$+\int\int_\Omega \alpha v_x^2+\gamma v_y^2dxdy, \tag{2.14}$$

where

$$\alpha=K\left(\frac{c_y-b_x}{2}-a\right)+\frac{1}{2}b+\frac{1}{2}K_yc$$
$$=\left(\frac{m}{2}-\mu-\delta\right)x+\delta y^2$$

and

$$\gamma=-a-\frac{c_y}{2}+\frac{b_x}{2}=M-\frac{\mu-m}{2}=m-2\mu-\delta>\mu-\delta.$$

On the *elliptic* region Ω^+, $K>0$ and

$$\left(\frac{m}{2}-\mu-\delta\right)x>\left(\frac{\mu}{2}-\delta\right)x\geq\delta x$$

provided we choose δ so small that $\mu/4\geq\delta$. Then on Ω^+,

$$\alpha\geq\delta\left(x+y^2\right)\geq\delta\left(x-y^2\right)=\delta K=\delta|K|.$$

On the *hyperbolic* region Ω^-, $K<0$ and

$$\alpha=\left(\frac{m}{2}-\mu\right)x+\delta\left(y^2-x\right)\geq\frac{\mu}{2}x+\delta\left(-K\right)\geq\delta|K|.$$

We show that the integrand of the boundary integral in (2.14) is non-negative: Because $v\in C^0\left(\overline{\Omega}\right)\cap H_0^1\left(\Omega;K\right)$,

$$|K|v_x^2+v_y^2=0$$

in a sufficiently small neighborhood of $\partial\Omega$. This implies that $v_x^2=v_y^2=0$ in a sufficiently small neighborhood of $\partial\Omega\backslash\left\{K=0\right\}$. That in turn implies that

$$Kv_x^2+v_y^2=0$$

in a sufficiently small neighborhood of $\partial\Omega\backslash\left\{K=0\right\}$. On the curve $K=0$, we have $x=y^2$ and $dx=2ydy$, implying that

$$cdx-bdy=\left(2\mu y^2-mx\right)dy=\left(2\mu-m\right)xdy.$$

But $m > 3\mu > 2\mu$, $x \geq 0$ on $\overline{\Omega}$, and $dy \leq 0$ on K; so on the resonance curve,

$$v_y^2 \left(cdx - bdy \right) \geq 0.$$

Thus the integrand of the boundary integral in (2.14) is bounded below by zero.

We find that if δ is sufficiently small relative to μ, then

$$(v, LHv) \geq \delta \int \int_\Omega \left(|K| v_x^2 + v_y^2 \right) dxdy. \tag{2.15}$$

The upper estimate is immediate, as

$$(v, LHv) = (v, Lu) \leq \|v\|_{H_0^1(\Omega;K)} \|Lu\|_{H^{-1}(\Omega;K)} . \tag{2.16}$$

Combining (2.15) and (2.16), we obtain

$$\|v\|_{H_0^1(\Omega;K)} \leq C \|Lu\|_{H^{-1}(\Omega;K)} . \tag{2.17}$$

The assertion of Lemma 2.1 now follows from (2.7) by the continuity of H as a map from $H_0^1 (\Omega; K)$ into $L^2 (\Omega; |K|)$. This completes the proof of Lemma 2.1.

Theorem 2.2. *Let Ω be star-shaped with respect to the flow of the vector field $-V = (mx, \mu y)$, where m and μ are defined as in Lemma 2.1. Suppose that x is nonnegative on $\overline{\Omega}$ and that the origin of coordinates lies on $\partial\Omega$. Then for every $f \in L^2 (\Omega; |K|^{-1})$ there is a unique weak solution $u \in H_0^1 (\Omega; K)$ to the Dirichlet problem (2.3), (2.4) where $K = x - y^2$.*

Proof. The proof follows the outline of the arguments in [17], Sec. 3. Defining a linear functional J_f by the formula

$$J_f (L\xi) = (f, \xi), \ \xi \in C_0^\infty (\Omega),$$

we estimate

$$|J_f (L\xi)| \leq ||f||_{L^2(\Omega;|K|^{-1})} ||\xi||_{L^2(\Omega;|K|)} \leq C ||f||_{L^2(\Omega;|K|^{-1})} ||L\xi||_{H^{-1}(\Omega;K)},$$

using Lemma 2.1. Thus J_f is a bounded linear functional on the subspace of $H^{-1} (\Omega; K)$ consisting of elements having the form $L\xi$ with $\xi \in C_0^\infty (\Omega)$. Extending J_f to the closure of this subspace by Hahn-Banach arguments,

the Riesz Representation Theorem guarantees the existence of an element $u \in H_0^1(\Omega; K)$ for which

$$\langle u, L\xi \rangle = (f, \xi),$$

where $\xi \in H_0^1(\Omega; K)$. There exists a unique, continuous, self-adjoint extension $L : H_0^1(\Omega; K) \to H^{-1}(\Omega; K)$. As a result, if a sequence u_n of smooth, compactly supported functions approximates u in the norm $H_0^1(\Omega; K)$, then Lu_n converges in norm to an element $\tilde{f}$ of $H^{-1}(\Omega; K)$. Taking the limit

$$\lim_{n\to\infty} \langle u - u_n, L\xi \rangle = \left(f - \tilde{f}, \xi \right),$$

we conclude that, because the left-hand side vanishes for all $\xi \in H_0^1(\Omega; K)$, the right-hand side must vanish as well. This proves the existence of a weak solution. Taking the difference of two weak solutions, we find that this difference is zero in $H_0^1(\Omega; K)$ by Lemma 2.1, the linearity of L, and the weighted Poincaré inequality [17]. This completes the proof of Theorem 2.2.

The unique-existence proofs of this section use estimates similar to those used in the proofs in Sec. 4 of [24] for existence alone. But the likelihood that the estimates would turn out to be similar appeared to be small on the basis of previous literature, and is rather surprising. In Sec. 5.1 of [24] it is shown that the estimates used to prove weak existence do not extend in an obvious way to proofs of uniqueness for the case $K = x$, in which the resonance curve is collinear with the flux line. Based on the physical discussion on p. 42 of [39], the collinear case would appear to be simpler than the case treated here, in which the two curves are tangent at an isolated point. But the simplicity of the case $K = x$ arises largely from the fact that in this case the plasma behaves like a perpendicularly stratified medium, in which wave motion satisfies an ordinary differential equation; *c.f.* [13]. The cautionary example of [24], Sec. 5.1, which suggests the difficulty of modifying the weak-existence methods to prove uniqueness in the case $K = x$, happens to fail in the case $K = x - y^2$. As we have shown in this section, a modification of the weak existence estimates will in fact lead to a uniqueness proof for weak solutions to (2.3), (2.4) for our choice of K. The case $K = x$ remains interesting from several points of view, and we will return to its study in Sec. 3.1.

The restriction that the points of Ω may not lie in the negative half-plane corresponds a requirement that boundary conditions be placed on one side of the flux line. As the resonance frequency is naturally restricted to the same

side of that line, the requirement seems to be compatible with the physical model.

3 Strong solutions to open boundary value problems

In this section we seek conditions sufficient for the existence of strong solutions in the cold plasma model. The conditions that we find will turn out to be extremely restrictive, but they are satisfied in a well known special case.

Consider a system of the form

$$Lu = \mathbf{f} \tag{3.1}$$

for an unknown vector

$$\mathbf{u} = \left(u_1\left(x,y\right), u_2\left(x,y\right)\right),$$

and a known vector

$$\mathbf{f} = \left(f_1\left(x,y\right), f_2\left(x,y\right)\right),$$

where $(x,y) \in \Omega \subset \mathbb{R}^2$. The operator L satisfies

$$\left(L\mathbf{u}\right)_1 = K\left(x,y\right)u_{1x} + u_{2y} + \text{ zeroth-order terms,} \tag{3.2}$$

$$\left(L\mathbf{u}\right)_2 = u_{1y} - u_{2x}. \tag{3.3}$$

As in the preceding section, $K\left(x,y\right)$ is continuously differentiable, negative on Ω^-, positive on Ω^+, and zero on a parabolic region (the resonance curve) separating the elliptic and hyperbolic regions. If $\left(f_1, f_2\right) = \left(f, 0\right)$, the components of the vector $\mathbf{u}$ are continuously differentiable, and $u_1 = u_x$, $u_2 = u_y$ for some twice-differentiable function $u(x,y)$, then the first-order system (3.1)-(3.3) reduces to a second-order scalar equation such as (2.3). Because the emphasis in this section is on the form of the boundary conditions, the presence or absence of zeroth-order terms will not affect the arguments provided the resulting system is symmetric positive.

We say that a vector $\mathbf{u} = (u_1, u_2)$ is in L^2 if each of its components is square-integrable. Such an object is a *strong solution* of an operator equation of the form (3.1), with given boundary conditions, if there exists a sequence

$\mathbf{u}^\nu$ of continuously differentiable vectors, satisfying the boundary conditions, for which $\mathbf{u}^\nu$ converges to $\mathbf{u}$ in L^2 and $L\mathbf{u}^\nu$ converges to $\mathbf{f}$ in L^2.

Sufficient conditions for a vector to be a strong solution were formulated by Friedrichs [8]. An operator L associated to an equation of the form

$$L\mathbf{u} = A^1\mathbf{u}_x + A^2\mathbf{u}_y + B\mathbf{u}, \tag{3.4}$$

where A^1, A^2, and B are matrices, is said to be *symmetric positive* if the matrices A^1 and A^2 are symmetric and the matrix

$$Q \equiv B^* - \frac{1}{2}\left(A^1_x + A^2_y\right)$$

is positive-definite, where B^* is the symmetrization of the matrix B :

$$B^* = \frac{1}{2}\left(B + B^T\right).$$

The differential equation associated to a symmetric positive operator is also said to be symmetric positive.

Boundary conditions for a symmetric positive equation can be given in terms of a matrix

$$\beta = n_1 A^1_{|\partial\Omega} + n_2 A^2_{|\partial\Omega}, \tag{3.5}$$

where (n_1, n_2) are the components of the outward-pointing normal vector on $\partial\Omega$. The boundary is assumed to be twice-continuously differentiable. Denote by $\mathcal{V}$ the vector space identified with the range of $\mathbf{u}$ in the sense that, considered as a mapping, we have $\mathbf{u} : \Omega \cup \partial\Omega \to \mathcal{V}$. Let $\mathcal{N}(\tilde{x}, \tilde{y})$, $(\tilde{x}, \tilde{y}) \in \partial\Omega$, be a linear subspace of $\mathcal{V}$ and let $\mathcal{N}(\tilde{x}, \tilde{y})$ depend smoothly on $\tilde{x}$ and $\tilde{y}$. A boundary condition $u \in \mathcal{N}$ is *admissible* if $\mathcal{N}$ is a maximal subspace of $\mathcal{V}$ with respect to non-negativity of the quadratic form $(\mathbf{u}, \beta\mathbf{u})$ on the boundary.

A set of sufficient conditions for admissibility is the existence of a decomposition ([8], Sec. 5)

$$\beta = \beta_+ + \beta_-, \tag{3.6}$$

for which: the direct sum of the null spaces for β_+ and β_- spans the restriction of $\mathcal{V}$ to the boundary; the ranges $\mathfrak{R}_\pm$ of $\beta_\pm$ have only the vector $\mathbf{u} = 0$ in common; and the matrix $\mu = \beta_+ - \beta_-$ satisfies

$$\mu^* = \frac{\mu + \mu^T}{2} \geq 0. \tag{3.7}$$

These conditions imply that the boundary condition

$$\beta_{-}\mathbf{u} = 0 \text{ on } \partial\Omega \tag{3.8}$$

is admissible for eq. (3.1) and the boundary condition

$$\mathbf{w}^T \beta_{+}^T = 0 \text{ on } \partial\Omega \tag{3.9}$$

is admissible for the adjoint problem

$$L^*\mathbf{w} = \mathbf{g} \text{ in } \Omega.$$

The linearity of the operator L and the admissibility conditions on the matrices $\beta_{\pm}$ imply that both problems possess unique, strong solutions.

Boundary conditions are *semi-admissible* if they satisfy properties (3.7) and (3.8). If $\mathbf{f}$ is in $L^2(\Omega)$ and (3.1) is a symmetric positive equation having semi-admissible boundary conditions, then (3.1) possesses a weak solution in the ordinary sense: a vector $\mathbf{u} \in L^2(\Omega)$ such that

$$\int_\Omega (L^*\mathbf{w}) \cdot \mathbf{u} \, d\Omega = \int_\Omega \mathbf{w} \cdot \mathbf{f} \, d\Omega$$

for all vectors $\mathbf{w}$ having continuously differentiable components and satisfying (3.9) ([8], Theorem 4.1).

Writing the higher-order terms of eqs. (3.2), (3.3) in the form

$$L\mathbf{u} = \begin{pmatrix} K(x,y) & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_y, \tag{3.10}$$

we will derive admissible boundary conditions for the system (3.1)-(3.3).

Slightly generalizing the type-change function of Sec. 2, we choose $K(x,y) = x - \sigma(y)$, where $\sigma(y) \geq 0$ is a continuously differentiable function of its argument satisfying (*c.f.* [23])

$$\sigma(0) = \sigma'(0) = 0, \tag{3.11}$$

$$\sigma'(y) \geq 0 \, \forall y \geq 0, \tag{3.12}$$

and

$$\sigma'(y) \leq 0 \, \forall y \leq 0. \tag{3.13}$$

Let the operator L in (3.1) be given by

$$(L\mathbf{u})_1 = [x - \sigma(y)]\, u_{1x} + u_{2y} + \kappa_1 u_1 + \kappa_2 u_2, \tag{3.14}$$

$$(L\mathbf{u})_2 = u_{1y} - u_{2x}, \tag{3.15}$$

where κ_1 and κ_2 are constants. We note that equations of this kind satisfy all the physical requirements that originally led to the selection of eq. (1.1) in [39] as a model for cold plasma waves.

By the *elliptic* portion $\partial\Omega^+$ of the boundary we mean points $(\tilde{x}, \tilde{y})$ of the domain boundary on which the type-change function $K(\tilde{x}, \tilde{y})$ is positive and by the *hyperbolic* portion $\partial\Omega^-$, boundary points for which the type-change function is negative. The *sonic* portion of the boundary consists of boundary points on which the type-change function vanishes.

In this section we prove a revision and extension of [24], Theorem 9:

Theorem 3.1. *Let Ω be a bounded, connected domain of $\mathbb{R}^2$ having C^2 boundary $\partial\Omega$. Let $\partial\Omega_1^+$ be a (possibly empty and not necessarily proper) subset of $\partial\Omega^+$. Let inequality (2.6) be satisfied on $\partial\Omega^+ \backslash \partial\Omega_1^+$. On $\partial\Omega_1^+$ let*

$$bn_1 + cn_2 \le 0 \tag{3.16}$$

and on $\partial\Omega \backslash \partial\Omega^+$, let

$$-bn_1 + cn_2 \ge 0. \tag{3.17}$$

Let $b(x,y)$ and $c(x,y)$ satisfy

$$b^2 + c^2 K \ne 0 \tag{3.18}$$

on Ω, with neither b nor c vanishing on Ω^+, and the inequalities:

$$2b\kappa_1 - b_x K - b + c_y K - c\sigma'(y) > 0 \text{ in } \Omega; \tag{3.19}$$

$$\begin{aligned}(2b\kappa_1 - b_x K - b + c_y K - c\sigma'(y))\,(2c\kappa_2 + b_x - c_y)& \\ - (b\kappa_2 + c\kappa_1 - c_x K - c - b_y)^2 > 0 \text{ in } \Omega;&\end{aligned} \tag{3.20}$$

$$K\,(bn_1 - cn_2)^2 + (cKn_1 + bn_2)^2 \le 0 \text{ on } \partial\Omega \backslash \partial\Omega^+. \tag{3.21}$$

Let L be given by (3.14), (3.15). Let the Dirichlet condition

$$-u_1 n_2 + u_2 n_1 = 0 \tag{3.22}$$

be satisfied on $\partial\Omega^+ \backslash \partial\Omega_1^+$ and let the Neumann condition

$$Ku_1 n_1 + u_2 n_2 = 0 \tag{3.23}$$

be satisfied on $\partial\Omega_1^+$. Then eqs. (3.1), (3.14), (3.15) possess a strong solution on Ω for every $\mathbf{f} \in L^2(\Omega)$.

Proof. Multiply both sides of eqs. (3.1), (3.14), (3.15) by the matrix

$$E = \begin{pmatrix} b & -cK \\ c & b \end{pmatrix}. \tag{3.24}$$

Condition (3.18) implies that E is invertible on Ω, and conditions (3.19), (3.20) imply that the resulting system is symmetric positive.

For all points $(\tilde{x}, \tilde{y}) \in \partial\Omega$, decompose the matrix

$$\beta\left(\tilde{x}, \tilde{y}\right) = \begin{pmatrix} K\left(bn_1 - cn_2\right) & cKn_1 + bn_2 \\ cKn_1 + bn_2 & -\left(bn_1 - cn_2\right) \end{pmatrix}$$

into a matrix sum having the form $\beta = \beta_+ + \beta_-$.

On $\partial\Omega^+ \backslash \partial\Omega_1^+$, decompose β into the submatrices

$$\beta_+ = \begin{pmatrix} Kbn_1 & bn_2 \\ Kcn_1 & cn_2 \end{pmatrix}$$

and

$$\beta_- = \begin{pmatrix} -Kcn_2 & Kcn_1 \\ bn_2 & -bn_1 \end{pmatrix}.$$

Then $\beta_- \mathbf{u} = 0$ under boundary condition (3.22). We have

$$\mu^* = (bn_1 + cn_2) \begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix},$$

so condition (2.6) implies that the Dirichlet condition (3.22) is semi-admissible on $\partial\Omega \backslash \partial\Omega_1^+$.

On $\partial\Omega_1^+$, choose

$$\beta_+ = \begin{pmatrix} -Kcn_2 & Kcn_1 \\ bn_2 & -bn_1 \end{pmatrix}$$

and

$$\beta_- = \begin{pmatrix} Kbn_1 & bn_2 \\ Kcn_1 & cn_2 \end{pmatrix}.$$

Then $\beta_- \mathbf{u} = 0$ under the Neumann boundary condition (3.23), and

$$\mu^* = -\left(bn_1 + cn_2\right)\begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix}$$

is positive semi-definite under condition (3.16).

On $\partial\Omega\backslash\partial\Omega^+$, choose $\beta_+ = \beta$ and take β_- to be the zero matrix. Then $\mu = \mu^* = \beta$ and

$$\mu_{11} = K\left(bn_1 - cn_2\right).$$

Because μ_{11} is non-negative by (3.17), μ^* is positive semi-definite by inequality (3.21), and no conditions need be imposed outside the elliptic portion of the boundary.

This yields semi-admissibility. We now prove admissibility.

On $\partial\Omega^+\backslash\partial\Omega_1^+$ the null space of β_- is composed of vectors satisfying the Dirichlet condition (3.22), which is imposed on that boundary arc. The null space of β_+ is composed of vectors satisfying the adjoint condition (3.23). On $\partial\Omega_1^+$, this relation is reversed. In order to show that the direct sum of these null spaces spans the two-dimensional space $\mathcal{V}_{|\partial\Omega^+}$, it is sufficient to show that the set

$$\left\{\begin{pmatrix} 1 \\ n_2/n_1 \end{pmatrix}, \begin{pmatrix} 1 \\ -Kn_1/n_2 \end{pmatrix}\right\}$$

is linearly independent there. Setting

$$c_1\begin{pmatrix} 1 \\ n_2/n_1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -Kn_1/n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

we find that $c_1 = -c_2$ and

$$-c_2\left(\frac{n_2^2 + Kn_1^2}{n_1 n_2}\right) = 0. \tag{3.25}$$

Equation (3.25) can only be satisfied on the elliptic boundary if $c_2 = 0$, implying that $c_1 = 0$. Thus the direct sum of the null spaces of $\beta_\pm$ on $\partial\Omega^+$ is linearly independent and must span $\mathcal{V}$ over that portion of the boundary.

On $\partial\Omega\backslash\partial\Omega^+$, the null space of β_- contains every 2-vector and the null space of β_+ contains only the zero vector; so on that boundary arc, their direct sum spans $\mathcal{V}$.

On $\partial\Omega^+\backslash\partial\Omega_1^+$, the range $\mathfrak{R}_+$ of β_+ is the subset of the range $\mathfrak{R}$ of β for which

$$v_2 n_1 - v_1 n_2 = 0 \tag{3.26}$$

for $(v_1, v_2) \in \mathcal{V}$; the range $\mathfrak{R}_-$ of β_- is the subset of $\mathfrak{R}$ for which

$$Kv_1n_1 + v_2n_2 = 0 \tag{3.27}$$

for $(v_1, v_2) \in \mathcal{V}$. Analogous assertions hold on $\partial\Omega_1^+$, in which the ranges of $\mathfrak{R}_+$ and $\mathfrak{R}_-$ are interchanged. Because if n_1 and n_2 are not simultaneously zero the system (3.26), (3.27) has only the trivial solution $v_2 = v_1 = 0$ on $\partial\Omega^+$, we conclude that $\mathfrak{R}_+ \cap \mathfrak{R}_- = \{0\}$ on $\partial\Omega^+$.

On $\partial\Omega\backslash\partial\Omega^+$, $\mathfrak{R}_- = \{0\}$, so $\mathfrak{R}_+ \cap \mathfrak{R}_- = \{0\}$ trivially.

The invertibility of E under condition (3.18) completes the proof of Theorem 3.1.

Remarks. *i*) By taking $\partial\Omega_1^+$ to be either the empty set or all of $\partial\Omega^+$, Theorem 3.1 implies the existence of strong solutions for either the open Dirichlet problem or the open Neumann problem. Because only the open Dirichlet problem was considered in Theorem 9 of [24], Theorem 3.1 of this report extends that result to the open cases of the Neumann and mixed Dirichlet-Neumann problems.

ii) A misprint in eq. (45) of [24] has been corrected in eq. (3.14). In Theorem 3.1, condition (3.18) has been added to the list of hypotheses in Theorem 9 of [24], the redundant condition (57) removed, and an error in eq. (59) corrected by eq. (3.21) of this report. The assumption that the boundary is piecewise smooth, which was default hypothesis in [24], seems to be too weak in general for strong solutions; see, however, [12], [14], and [31]-[34].

iii) Only conditions (3.19) and (3.20) have anything to do with the cold plasma model. Otherwise, Theorem 3.1 is about interpreting Friedrichs' theory in the context of a collection of boundary arcs which are starlike with respect to a corresponding collection of vector fields. For example, the argument leading to eq. (3.25) suggests that the Tricomi problem is strongly ill-posed under the hypotheses of the theorem, whatever the type-change function K. This is because in the Tricomi problem, data are given on both the elliptic boundary and a characteristic curve; but on characteristic curves, K satisfies

$$K = -\frac{n_2^2}{n_1^2}. \tag{3.28}$$

Substituting this equation into eq. (3.25), we find that the equation is satisfied on characteristic curves without requiring the constants c_1 and c_2 to be zero.

However, the theorem is less restrictive if the operator in (3.1) is given by

$$(L\mathbf{u})_1 = [x - \sigma(y)]\, u_{1x} - u_{2y} + \kappa_1 u_1 + \kappa_2 u_2,$$
$$(L\mathbf{u})_2 = -u_{1y} + u_{2x}, \tag{3.29}$$

where, again, κ_1 and κ_2 are constants. This variant also arises in the cold plasma model (see [22] and [30]) and is analogous to the variant of the Tricomi equation,

$$yu_{xx} - u_{yy} = 0,$$

studied in various contexts by Friedrichs [8], Katsanis [11], Sorokina [33], [34], and Didenko [7]. In that case, choose

$$E = \begin{pmatrix} b & cK \\ c & b \end{pmatrix}.$$

Obvious modifications of conditions (3.19) and (3.20) guarantee that the equation

$$EL\mathbf{u} = E\mathbf{f}$$

will be symmetric positive. Condition (3.18) must be replaced by the invertibility condition

$$b^2 - c^2 K \neq 0,$$

which is restrictive on the subdomain Ω^+ rather than on Ω^- as in (3.18). Most importantly, the discussion leading to Table 1 of [11] now applies, with only minor changes, and one can obtain a long list of possible starlike boundaries which result in strong solutions to suitably formulated problems of Dirichlet or Neumann type. In particular, one can formulate a Tricomi problem which is strongly well-posed.

iv) The hypotheses of Theorem 3.1 have a rather formal appearance. We expect them to be harsh, as the known singularity at the origin should dramatically restrict the kinds of smoothness results that we can prove. But many of the conditions have natural interpretations. For example, inequalities (2.6), (3.16), and (3.17) are satisfied whenever boundary arcs are starlike with respect to an appropriate vector field, and (3.21) is always satisfied on the characteristic boundary:

Proposition 3.2. *Let Γ be a characteristic curve for eq. (3.1), with the higher-order terms of the operator L satisfying (3.10). Then the left-hand side of inequality (3.21) is identically zero on Γ.*

Proof. We have, using eq. (3.28),

$$(cKn_1 + bn_2)^2 = c^2K^2n_1^2 + 2Kcbn_1n_2 + b^2n_2^2$$

$$= -c^2K^2\frac{n_2^2}{K} + 2Kcbn_1n_2 - b^2Kn_1^2 = -K\left(c^2n_2^2 - 2cbn_1n_2 + b^2n_1^2\right)$$

$$= -K\left(cn_2 - bn_1\right)^2.$$

Substituting the extreme right-hand side of this equation into the second term of (3.21) completes the proof.

3.1 An explicit example

A simple example which illustrates the hypotheses of Theorem 3.1 can be constructed for the special case $\sigma(y) \equiv 0$. In that case the system (3.1), (3.14), (3.15) can be reduced, by taking $u_1 = u_x$, $u_2 = u_y$, and $\mathbf{f} = 0$, to the *Cinquini-Cibrario equation* [5]

$$xu_{xx} + u_{yy} + \text{ lower-order terms } = 0. \tag{3.30}$$

In the context of the cold plasma model, this case corresponds to a resonance curve which is collinear with a flux line. In such situations eq. (1.1) can be replaced by an ordinary differential equation, as was discussed at the end of Sec. 2, so this choice is of little direct interest for the cold plasma model. But the Cinquini-Cibrario equation is interesting in its own right, in connection with normal forms for second-order linear elliptic-hyperbolic equations [3], [4]. Moreover, polar forms of eq. (3.30) arise in models of atmospheric and space plasmas − compare eq. (9) of [10], eq. (16) of [37], and eq. (19a) of [38], with Sec. 3 of [6]. (Such plasmas are cold, but in a relative rather than absolute sense.)

In addition to taking $\sigma(y) \equiv 0$, choose $\kappa_1 = 1$; $\kappa_2 = 0$; $b = x + M$, where M is a positive constant which is assumed to be large in comparison with all other parameters of the problem − in particular, $b > 0\,\forall x \in \overline{\Omega}$; $c = \epsilon y$, where ϵ is a small positive constant; $(n_1, n_2) = (dy/ds, -dx/ds)$, where s is arc length on the boundary. Inequalities (3.18)-(3.20) are satisfied for M sufficiently large.

Let the hyperbolic region Ω^- be bounded by intersecting characteristic curves originating on the sonic line. Condition (3.21) is satisfied on $\partial\Omega^-$ by Proposition 3.2. Condition (3.17) is satisfied for M sufficiently large. As a concrete example, let $\partial\Omega^- = \Gamma^- \cup \Gamma^+$, where

$$\Gamma^\pm = \left\{(x,y) \in \Omega^- | y = \pm 2\left(\sqrt{-x} - 2\right)\right\}.$$

These curves intersect at the point $(-4,0)$. Their intersection is not C^2, but it can be easily "smoothed out" (by the addition of a small noncharacteristic curve connecting the points $(-4+\delta_0, \pm\delta_1)$ for $0 < \delta_0, \delta_1 << 1$) without violating either of the governing inequalities. Let the elliptic boundary $\partial\Omega^+$ be a smooth convex curve, symmetric about the x-axis, with endpoints at $(0,\pm 4)$ on the sonic line. Let the disconnected subset $\partial\Omega_1^+$ of $\partial\Omega^+$ take the form of two small "smoothing curves," on which the slope of the tangent line to $\partial\Omega^+$ changes sign in order to prevent a cusp at the two endpoints. Inequality (2.6) is satisfied on $\partial\Omega^+ \backslash \Omega_1^+$ and, again assuming that M is sufficiently large, inequality (3.16) is satisfied on the two smoothing curves comprising $\partial\Omega_1^+$.

The domain Ω of this construction is identical to the domain illustrated in Fig. 2 on p. 277 of [6], except that the cusps at points R, M, and N of that figure are smoothed out in Ω near the points $(-4,0)$ and $(0,\pm 4)$.

Theorem 3.1 implies that strong solutions to a homogeneous mixed Dirichlet-Neumann problem for the first-order inhomogeneous form of the Cinquini-Cibrario equation exist on this natural class of domains for any square-integrable forcing function.

3.2 Symmetric positive operators on domains having multiply starlike boundaries

On the basis of the observations in Remarks *iii)* and *iv)*, we reformulate Theorem 3.1 for an arbitrary type-change function $-$ that is, for a smooth function $K(x,y)$ which is positive on all points in a subset Ω^+ of Ω, negative on all points of a subset Ω^- of Ω, and zero on a smooth curve $\mathfrak{K} \in \Omega$ separating Ω^+ and Ω^-, where

$$\Omega = \Omega^+ \cup \Omega^- \cup \mathfrak{K}.$$

The proof of Theorem 3.1 will also prove:

Corollary 3.3. *Let Ω be a bounded, connected domain of $\mathbb{R}^2$ having C^2 boundary $\partial\Omega$, oriented in a counterclockwise direction. Let $\partial\Omega_1^+$ be a (possibly empty and not necessarily proper) subset of $\partial\Omega^+$. Suppose that EL is a*

symmetric positive operator, where L satisfies (3.10) (with the possible addition of lower-order terms) and E satisfies (3.24) with condition (3.18). Let $\partial\Omega^+ \backslash \partial\Omega_1^+$ *be starlike with respect to the vector field* $V^+ = -\left(b(x,y), c(x,y)\right);$ *let* $\partial\Omega_1^+$ *be starlike with respect to the vector field* $V_1^+ = \left(b(x,y), c(x,y)\right);$ *let* $\partial\Omega \backslash \partial\Omega^+$ *be starlike with respect to the vector field* $V^- = \left(b(x,y), -c(x,y)\right).$ *Let the union* $\partial\Omega \backslash \partial\Omega^+$ *of the parabolic and hyperbolic boundaries be sub-characteristic in the sense of (3.21). Then the mixed boundary value problem given by eqs. (3.1)-(3.3), with condition (3.22) satisfied on* $\partial\Omega^+ \backslash \partial\Omega_1^+$ *and condition (3.23) satisfied on* $\partial\Omega_1^+$*, possesses a strong solution for every* $\mathbf{f} \in L^2(\Omega)$.

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References

[1] J. Barros-Neto and I. M. Gelfand, *Fundamental solutions for the Tricomi operator*, Duke Math. J. 98 (1999), pp. 465–483.

[2] Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, American Mathematical Society, Providence, 1968.

[3] A. V. Bitsadze, *Equations of the Mixed Type*, Pergammon Press, New York, 1964.

[4] M. Cibrario, Sulla riduzione a forma canonica delle equazioni lineari alle derivate parziali di secondo ordine di tipo misto, *Rendiconti del R. Insituto Lombardo* **65** (1932).

[5] M. Cibrario, *Alcuni teoremi di esistenza e di unicita per l'equazione* $xz_{xx} + z_{yy} = 0$, Atti R. Acc. Torino 68 (1932–1933).

[6] M. Cibrario, *Intorno ad una equazione lineare alle derivate parziali del secondo ordine di tipe misto iperbolico-ellittica*, Ann. Sc. Norm. Sup. Pisa, Cl. Sci., Ser. 2, 3, Nos. 3, 4 (1934), pp. 255–285.

[7] V. P. Didenko, *On the generalized solvability of the Tricomi problem*, Ukrain. Math. J. 25 (1973), pp. 10–18.

[8] K. O. Friedrichs, *Symmetric positive linear differential equations*, Commun. Pure Appl. Math. 11 (1958), pp. 333–418.

[9] V. L. Ginzburg, *Propagation of Electromagnetic Waves in Plasma*, translated from the Russian by J. B. Sykes and R. J. Tayler, Pergamon, New York, 1970.

[10] Janhunen P., *Magnetically dominated plasma models of ball lightning*, Annales Geophysicae. Atmospheres, Hydrospheres and Space Sciences 9, No. 6 (1991), pp. 377–380.

[11] T. Katsanis, *Numerical solution of Tricomi equation using theory of symmetric positive differential equations*, SIAM J. Numer. Anal. 6 (1969), pp. 236–253.

[12] P. D. Lax and R. S. Phillips, *Local boundary conditions for dissipative symmetric linear differential operators*, Commun. Pure Appl. Math. 13 (1960), pp. 427–455.

[13] E. Lazzaro and C. Maroli, *Lower hybrid resonance in an inhomogeneous cold and collisionless plasma slab*, Nuovo Cim. 16B (1973), no. 1, pp. 44–54.

[14] C. S. Lin, *The local isometric embedding in* $\mathbb{R}^3$ *of a 2-dimensional Riemanian manifolds with Gaussian curvature changing sign cleanly*, Commun. Pure Appl. Math. 39 (1986), pp. 867–887.

[15] *D. Lupo and K. R. Payne, A dual variational approach to a class of nonlocal semilinear Tricomi problems*, Nonlinear Differential Equations Appl. 6, no. 3 (1999), pp. 247–266.

[16] D. Lupo and K. R. Payne, *Critical exponents for semilinear equations of mixed elliptic-hyperbolic and degenerate types*, Commun. Pure Appl. Math. 56 (2003), pp. 403–424.

[17] D. Lupo, C. S. Morawetz, and K. R. Payne, *On closed boundary value problems for equations of mixed elliptic-hyperbolic type*, Commun. Pure Appl. Math. 60 (2007), pp. 1319–1348.

[18] D. Lupo, C. S. Morawetz, and K. R. Payne, *Erratum: "On closed boundary value problems for equations of mixed elliptic-hyperbolic type," [Commun. Pure Appl. Math.* ***60*** *1319-1348 (2007)]* Commun. Pure Appl. Math. 61 (2008), p. 594.

[19] A. R. Manwell, *The Tricomi Equation with Applications to the Theory of Plane Transonic Flow*, Pitman Publishing Company, LTD, London, 1979.

[20] C. S. Morawetz, *A weak solution for a system of equations of elliptic-hyperbolic type*, Commun. Pure Appl. Math. 11 (1958), pp. 315-331 (1958).

[21] C. S. Morawetz, *The Dirichlet problem for the Tricomi equation*, Commun. Pure Appl. Math. 23 (1970), pp. 587–601.

[22] C. S. Morawetz, D. C. Stevens, and H. Weitzner, *A numerical experiment on a second-order partial differential equation of mixed type*, Commun. Pure Appl. Math. 44 (1991), pp. 1091–1106.

[23] T. H. Otway, *A boundary-value problem for cold plasma dynamics*, J. Appl. Math. 3, no. 1 (2003), pp. 17–33.

[24] T. H. Otway, *Energy inequalities for a model of wave propagation in cold plasma*, Publ. Mat. 52 (2008), pp. 195–234.

[25] T. H. Otway, *Variational equations on mixed Riemannian-Lorentzian metrics*, J. Geom. Phys. 58 (2008), pp. 1043–1061.

[26] T. H. Otway, *Mathematical aspects of the cold plasma model*, in: *Perspectives in Mathematical Sciences*, J. Duan, X. Fu, and Y. Yang, eds., World Scientific Press, Singapore, 2010, pp. 181-210.

[27] K. R. Payne, *Interior regularity of the Dirichlet problem for the Tricomi equation*, J. Math. Anal. Appl. 199 (1996), pp. 271–292.

[28] K. R. Payne, *Boundary geometry and location of singularities for solutions to the Dirichlet problem for Tricomi type equations*, Houston J. Math. 23 (1997), pp. 709–731.

[29] M. Pilant, *The Neumann problem for an equation of Lavrent'ev-Bitsadze type*, J. Math. Anal. Appl. 106 (1985), pp. 321–359.

[30] A. D. Piliya and V. I. Fedorov, *Singularities of the field of an electromagnetic wave in a cold anisotropic plasma with two-dimensional inhomogeneity*, Sov. Phys. JETP 33 (1971), pp. 210–215.

[31] L. Sarason, *On weak and strong solutions of boundary value problems*, Commun. Pure Appl. Math. 15 (1962), pp. 237–288.

[32] L. Sarason, *Elliptic regularization for symmetric positive systems*, J. Math. Mech. 16, No. 8 (1967), pp. 807–827.

[33] N. G. Sorokina, *Strong solvability of the Tricomi problem* (in Russian), Ukrain. Mat. Zh. 18, No. 6 (1966), pp. 65–77.

[34] N. G. Sorokina, *The strong solvability of the generalized Tricomi problem*, Ukrainian Math. J. 24 (1973), pp. 558–561.

[35] T. H. Stix, *The Theory of Plasma Waves*, McGraw-Hill, New York, 1962.

[36] D. G. Swanson, *Plasma Waves*, 2nd edition, Institute of Physics, Bristol, 2003.

[37] K. H. Tsui, *A self-similar magnetohydrodynamic model for ball lightnings*, Phys. of Plasmas 13 (2006), 072102.

[38] K.H. Tsui and A. Serbeto, *Time-dependent magnetohydrodynamic self-similar extragalactic jets*, Astrophysical J. 658 (2007), pp. 794–803.

[39] H. Weitzner, "Wave propagation in a plasma based on the cold plasma model," Courant Inst. Math. Sci. Magneto-Fluid Dynamics Div. Report MF-103, August, 1984.

[40] H. Weitzner, *Lower hybrid waves in the cold plasma model*, Commun. Pure Appl. Math. 38 (1985), pp. 919–932.

[41] Y. Yamamoto, "Existence and uniqueness of a generalized solution for a system of equations of mixed type," Ph.D. Dissertation, Polytechnic University of New York, 1994.